On the number of points over finite fields on varieties related to cluster algebras

F. Chapoton

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Abstract

We compute the number of points over finite fields of some algebraic varieties related to cluster algebras of finite type. More precisely, these varieties are the fibers of the projection map from the cluster variety to the affine space of coefficients.

1 Introduction

Cluster algebras have been introduced by S. Fomin and A. Zelevinsky around 2000 [FZ02, FZ03, BFZ05, FZ07], and have since been a very active subject. Many connections have been found to combinatorics [FST08], Poisson geometry [FG03] and representation theory [BMR+06]. Cluster algebras are commutative algebras, and can therefore be considered as objects of algebraic geometry. In this article, we consider some algebraic varieties closely related to the spectrum of cluster algebras. As a first step towards computing their cohomology, we count their points over finite fields.

More precisely, we use as a starting point a theorem [BFZ05, Corollary 1.17] which gives a presentation by generators and relation of acyclic cluster algebras. We use this presentation to define, for each tree, a family of algebraic varieties depending on parameters in an affine space.

In the classical cluster algebra setting, this affine space of parameters corresponds to the so-called coefficients variables in cluster algebras. The fiber over a point in our affine space of parameters is a fiber of the map from the spectrum of the cluster algebra to the spectrum of its coefficient ring.

For simplicity, we restrict ourself to simply-laced cluster algebras of finite type, which are indexed by the usual $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ list of Dynkin diagrams. We have implicitly chosen to work with the alternating orientation of these Dynkin diagrams, but the results do not depend on the orientation.

We are also able to obtain more precise results in the case of type $\mathbb{A}$. Here, we describe the cohomology with compact support, in the even case.

The interest of the results may be in the very simple shape of the answers in the generic case (formulas (6), (7) in type $\mathbb{A}$ and formulas (22), (24) in type $\mathbb{D}$), which are nice polynomials in the cardinal q of the finite field $\mathbb{F}_q$. It would be very interesting to see if the cohomology with compact support is as simple as one may expect from these nice polynomials. We prove that this is indeed the case in type $\mathbb{A}_n$ with n even.

2 General results

In this section, we introduce general definitions and tools valid for all trees. Later on, we will use these for finite-type simply-laced Dynkin diagrams.

2.1 Definition

Let T be a tree, *i.e.* a finite graph which is connected and simply connected. We will write $s - t$ to denote that s and t are adjacent vertices of T.

Let $\boldsymbol{\alpha} = (\alpha_t)_{t \in T}$ be a function on the set of vertices of T with values in some field $\mathbb{K}$.

Let us call $X_T(\boldsymbol{\alpha})$ the affine scheme over $\mathbb{K}$ defined by

$$x_t x'_t = 1 + \alpha_t \prod_{s-t} x_s, \tag{1}$$

for all vertices t of T.

One can also consider this set of equations as defining a family of schemes over the base affine scheme $\operatorname{Spec} \mathbb{Z}[(\alpha_t)_{t \in T}]$. We will study the fibers of this family. From now on, we will assume (unless explicitly stated otherwise) that the α_t are invertible. This amounts to restrict the family to $\operatorname{Spec} \mathbb{Z}[(\alpha_t, \alpha_t^{-1})_{t \in T}]$.

Remark 2.1 *Instead of a tree T, one can also consider a disjoint union of trees F, in which case the variety $X_F(\boldsymbol{\alpha})$ will be the product of the varieties associated with the connected components of F.*

2.2 Reduction using domino tiling

Every tree T is a bipartite graph. Let us fix a choice of black and white vertices such that every edge has a white end and a black end.

Lemma 2.2 *Let s and t be adjacent vertices. Let $\boldsymbol{\beta}$ be the function defined by*

$$\begin{aligned} &\beta_s = 1, && \\ &\beta_u = \alpha_u / \alpha_s && \text{if } u \neq s \text{ and } u - t, \\ &\beta_u = \alpha_u && \text{else.} \end{aligned}$$

Then $X_T(\boldsymbol{\alpha})$ is isomorphic to $X_T(\boldsymbol{\beta})$.

Proof. One uses the change of variable $x_t = \alpha_s x_t$ and $x'_t = x'_t / \alpha_s$. ■

Note that this operation only changes the values of the function on vertices that have the same color as s. In a pictorial way, the value at vertex s jumps over vertex t and get spread over the other neighbors of t.

A **partial domino tiling** of T is a subset of the set of edges such that every vertex appears at most once among the ends of the chosen edges. It is called **full** if every vertex appears exactly once.

Lemma 2.3 *Let T be a tree with a full domino tiling. Then T has a white leaf.*

Proof. By an easy induction on the number of vertices. ■

Proposition 2.4 *Let T be a tree, endowed with a partial domino tiling. Every $X_T(\boldsymbol{\alpha})$ is isomorphic to some $X_T(\boldsymbol{\beta})$ where $\beta_s = 1$ for every vertex s which is covered by a domino.*

Proof. The proof uses only Lemma 2.2 to modify the function $\boldsymbol{\alpha}$. One can therefore treat black and white vertices separately. Let us just prove the statement for white vertices, the case of black vertices being the same with colors exchanged.

Let C be the set of vertices covered by dominoes. The partial domino tiling of T defines for every white vertex $s \in C$ a canonical black neighbor $B(s)$.

Let us orient the edges of T from white to black inside the dominoes (from s to $B(s)$) and from black to white outside the dominoes.

Then the induced subgraph of T on the set C of vertices covered by dominoes is a disjoint union of trees with oriented edges. One has to flip every white vertex $s \in C$ over the black vertex $B(s)$ (which means using Lemma 2.2 to replace $\boldsymbol{\alpha}$ by a modified function), in a well-chosen order. This order must be compatible with the partial order given by the orientation of edges : one has to start with white leaves (which exist by Lemma 2.3).

At the end of the process, one obtains a function $\boldsymbol{\beta}$ such that every white vertex in C has value 1 and such that $X_T(\boldsymbol{\alpha})$ is isomorphic to $X_T(\boldsymbol{\beta})$. ■

Remark 2.5 *For every tree, one can find a partial domino tiling where the only vertices which are not covered are leaves.*

2.3 Induction by removal of leaves

Let T be a tree and let f be a leaf of T. Let g be the unique vertex adjacent to f. Let $\boldsymbol{\alpha}$ be a function on T.

Let T' be the tree obtained from T by removing f. For every element β in the ground field $\mathbb{K}$, let $\boldsymbol{\alpha}'(\beta)$ be the function on T' defined by

$$\alpha'_g(\beta) = \alpha_g \beta, \tag{2}$$

$$\alpha'_s(\beta) = \alpha_s \text{ if } s \neq g. \tag{3}$$

Let T'' be the tree or disjoint union of trees obtained from T by removing f and g. Let $\boldsymbol{\alpha}''$ be the function on T'' defined by

$$\alpha''_s = -\alpha_s/\alpha_f \qquad \text{if } s - g \text{ in } T, \tag{4}$$

$$\alpha''_s = \alpha_s \qquad \text{else.} \tag{5}$$

Proposition 2.6 *The scheme $X_T(\boldsymbol{\alpha})$ is the disjoint union of the scheme $A_1 \times X_{T''}(\boldsymbol{\alpha}'')$ and of a variety fibered over $A_1 \setminus \{0\}$ with fiber $X_{T'}(\boldsymbol{\alpha}'(\beta))$ over β.*

Proof. One simply has to separate points according to whether $x_f = 0$ (which gives the first part) or not (which gives the variety fibered over $A_1 \setminus \{0\}$).

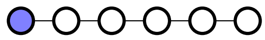
If one assumes $x_f = 0$, the equation (1) for vertex f gives the invertible value $-1/\alpha_f$ for x_g. Then the equation (1) for vertex g gives a value for x'_g. There remains a free variable x'_f and the equations for T'' with function $\boldsymbol{\alpha}''$. This corresponds to the product of the affine space A_1 and the variety $X_{T''}(\boldsymbol{\alpha}'')$

If one now assumes that x_f is invertible, then the equation (1) for vertex f gives a value to x'_f. One can remove this equation; there remains the equations

for T' with a function depending on the value of x_f. If one moreover fixes the value of x_f to be an invertible element β of $\mathbb{K}$, then one gets the equations of $X_{T'}(\boldsymbol{\alpha}'(\beta))$. ■

3 Type $\mathbb{A}$

We will now consider the Dynkin diagrams of type $\mathbb{A}$.



3.1 Number of points over finite fields

In type $\mathbb{A}_n$, the cluster algebra is generated by n cluster variables $x_1, \dots, x_n$ (which form a cluster), the n adjacent cluster variables $x'_1, \dots, x'_n$ and n coefficient variables $\alpha_1, \dots, \alpha_n$ with the following relations:

$$\begin{aligned} x_1 x'_1 &= 1 + \alpha_1 x_2, \\ x_2 x'_2 &= 1 + \alpha_2 x_1 x_3, \\ &\dots \\ x_{n-1} x'_{n-1} &= 1 + \alpha_{n-1} x_{n-2} x_n, \\ x_n x'_n &= 1 + \alpha_n x_{n-1}. \end{aligned}$$

Let us call $X_{\mathbb{A}_n}(\alpha_1, \dots, \alpha_n)$ this variety. As before, we assume that the α_i are invertible.

Proposition 3.1 *If n is even, then $X_{\mathbb{A}_n}(\alpha_1, \dots, \alpha_n) \simeq X_{\mathbb{A}_n}(1, 1, \dots, 1)$.*

If n is odd, then $X_{\mathbb{A}_n}(\alpha_1, \dots, \alpha_n) \simeq X_{\mathbb{A}_n}(\alpha, 1, \dots, 1)$, for some α depending only on the α_i with i odd.

Proof. This is obtained by applying Proposition 2.4 to the obvious full domino tiling (even case) or to the partial domino tiling avoiding only the first vertex (odd case). ■

For short, we will denote $X_n(\alpha)$ for $X_{\mathbb{A}_n}(\alpha, 1, \dots, 1)$. For the number of points of these varieties over finite fields, we will use the following notation: $N_{\mathbb{A}_n}(\alpha)$ is the number of points of $X_n(\alpha)$. When n is even, we will also use $N_{\mathbb{A}_n}$ for short.

Proposition 3.2 *If n is even, then*

$$N_{\mathbb{A}_n} = \frac{q^{n+2} - 1}{q^2 - 1}. \tag{6}$$

If n is odd and $\alpha \neq (-1)^{(n+1)/2}$, then

$$N_{\mathbb{A}_n}(\alpha) = \frac{(q^{(n+1)/2} - 1)(q^{(n+3)/2} - 1)}{q^2 - 1}. \tag{7}$$

If n is odd, then

$$N_{\mathbb{A}_n}((-1)^{(n+1)/2}) = \frac{(q^{(n+1)/2} - 1)(q^{(n+3)/2} - 1)}{q^2 - 1} + q^{(n+1)/2}. \tag{8}$$

Proof. By induction using leaf-removal (Proposition 2.6). The statement is clear if $n = 0$, in which case the variety $X_{\mathbb{A}_0}()$ is just a point. It is also immediate if $n = 1$.

Let us first note that the statement implies (for n even or odd) that

$$\sum_{\alpha \in \mathbb{F}_q^*} N_{\mathbb{A}_n}(\alpha) = \frac{q^{n+2} + (-1)^{n+1}}{q+1}. \tag{9}$$

Assume that n is even. Then Proposition 2.6 becomes

$$N_{\mathbb{A}_n} = q N_{\mathbb{A}_{n-2}} + \sum_{\alpha \in \mathbb{F}_q^*} N_{\mathbb{A}_{n-1}}(\alpha), \tag{10}$$

which can be rewritten (using (9)) as

$$N_{\mathbb{A}_n} = q N_{\mathbb{A}_{n-2}} + \frac{q^{n+1} + 1}{q+1}. \tag{11}$$

This implies the expected formula for $N_{\mathbb{A}_n}$.

Assume that n is odd. Then Proposition 2.6 becomes

$$N_{\mathbb{A}_n}(\alpha) = q N_{\mathbb{A}_{n-2}}(-1/\alpha) + (q-1) N_{\mathbb{A}_{n-1}}, \tag{12}$$

which can be rewritten as

$$N_{\mathbb{A}_n}(\alpha) = q N_{\mathbb{A}_{n-2}}(-1/\alpha) + \frac{q^{n+1} - 1}{q+1}. \tag{13}$$

Note that $\alpha = (-1)^{(n+1)/2}$ if and only if $-1/\alpha = (-1)^{(n-2+1)/2}$. The induction hypothesis then implies the expected formulas for $N_{\mathbb{A}_n}(\alpha)$. ■

Let $Y_{\mathbb{A}_n}$ be the union of all varieties $X_n(\alpha)$ for α invertible. By a natural convention, $Y_{\mathbb{A}_0}$ is just $A^1 \setminus \{0\}$. Let us note as a lemma the formula (9) that we have obtained in the proof of Prop. 3.2.

Lemma 3.3 *For every $n \geq 0$, one has*

$$\sum_{\alpha \in \mathbb{F}_q^*} N_{\mathbb{A}_n}(\alpha) = \frac{q^{n+2} + (-1)^{n+1}}{q+1}. \tag{14}$$

This is the number of points on $Y_{\mathbb{A}_n}$ over the finite field $\mathbb{F}_q$.

Let now $Z_{\mathbb{A}_n}$ be the union of all varieties $X_n(\alpha)$ for any α (we do not assume here that α is invertible).

Proposition 3.4 *For every $n \geq 1$, the space $Z_{\mathbb{A}_n}$ is the disjoint union of $Y_{\mathbb{A}_n}$ and $Y_{\mathbb{A}_{n-1}}$. The number of points on $Z_{\mathbb{A}_n}$ over the finite field $\mathbb{F}_q$ is q^{n+1}.*

Proof. The proof of this decomposition is obvious: either $\alpha = 0$ (and one obtains $Y_{\mathbb{A}_{n-1}}$) or not (in which case one gets $Y_{\mathbb{A}_n}$). The counting result follows from Lemma 3.3. ■

This result immediately suggests that $Z_{\mathbb{A}_n}$ may just be an affine space. This is proved in the next section and will allow to compute some cohomology groups.

We will use later the following result.

Lemma 3.5 *For every even $n \geq 1$, the space $Y_{\mathbb{A}_n}$ is isomorphic to the product of $X_n(1)$ with $A^1 \setminus \{0\}$.*

Proof. This is a simple consequence of Lemma 2.2. One has clearly a fibration, and this is made trivial by a simple change of variables. ■

3.2 Cohomology with compact supports

Proposition 3.6 *For every $n \geq 1$, there is a surjective morphism ϕ from $Z_{\mathbb{A}_{n+1}}$ to $Z_{\mathbb{A}_n}$ with fiber A^1. For every $n \geq 1$, there is an isomorphism $Z_{\mathbb{A}_n} \simeq A^{n+1}$.*

Proof. The morphism ϕ from $Z_{\mathbb{A}_{n+1}}$ to $Z_{\mathbb{A}_n}$ is defined by forgetting the first equation (1):

$$x_1 x_1' = 1 + \alpha x_2. \tag{15}$$

One simply has to shift down the indices of variables x_i and x_i' for $i \geq 2$ and let x_1 play the role of α.

As it is not possible that both x_1 and x_2 vanish (by the second equation (1)), the first equation is the equation of a line in the plane with coordinates x_1', α. Therefore every fiber of ϕ is a line.

One can easily check that $Z_{\mathbb{A}_1} \simeq A^2$. Then the expected isomorphism follows by induction. ■

Corollary 3.7 *As an open set of $Z_{\mathbb{A}_n}$, $Y_{\mathbb{A}_n}$ is smooth.*

Recall that the cohomology with compact support of the affine space A^n is very simple: the only non-zero group is $H_c^{2n}(A^n) \simeq \mathbb{Q}(n)$, where $\mathbb{Q}(n)$ is the Tate Hodge structure of weight n.

Proposition 3.8 *For $n \geq 0$, the non-zero cohomology groups with compact support of $Y_{\mathbb{A}_n}$ are*

$$H_c^{i+n+1}(Y_{\mathbb{A}_n}) \simeq \mathbb{Q}(i), \tag{16}$$

for $0 \leq i \leq n+1$.

Proof. The proof is by induction on n. The statement is true if $n = 0$. One then uses the long exact sequence in cohomology with compact support for the open-closed decomposition $Z_{\mathbb{A}_n} = Y_{\mathbb{A}_n} \sqcup Y_{\mathbb{A}_{n-1}}$ (see Prop. 3.4), together with Prop. 3.6. ■

One can then obtain the cohomology with compact support of $X_n(1)$ when n is even.

Proposition 3.9 *For $n \geq 0$ even, the non-zero cohomology groups with compact support of $X_n(1)$ are*

$$H_c^{i+n}(X_n(1)) \simeq \mathbb{Q}(i), \tag{17}$$

for all even i between 0 and n.

Proof. By induction on n. The statement is true for $n = 0$ with the natural convention that $X_{\mathbb{A}_0}(\,)$ is a point.

One uses two ingredients. The first one is the long exact sequence for the open-closed decomposition $X_n(1) = Y_{\mathbb{A}_{n-1}} \sqcup A^1 \times X_{n-2}$. The second one is the Künneth isomorphism describing the cohomology of the product $Y_{\mathbb{A}_n} \simeq A^1 \setminus \{0\} \times X_n(1)$ (see Lemma 3.5). We also need the fact that the cohomology with compact support of $A^1 \setminus \{0\}$ is $\mathbb{Q}(0)$ in degree 1 and $\mathbb{Q}(1)$ in degree 2.

From the long exact sequence, one gets exact sequences

$$0 \to \mathbb{Q}(i) \to H_c^{i+n}(X_n(1)) \to \mathbb{Q}(i+1) \to \mathbb{Q}(i+1) \to H_c^{i+n+1}(X_n(1)) \to 0, \tag{18}$$

for even i between 0 and n.

One would like to conclude that $H_c^{i+n}(X_{\mathbb{A}_{n+2}}) \simeq \mathbb{Q}(i)$ and $H_c^{i+n+1}(X_{\mathbb{A}_{n+2}}) \simeq 0$.

Assume on the contrary that, for some even i, $H_c^{i+n+1}(X_{\mathbb{A}_n}) \simeq \mathbb{Q}(i+1)$ and $H_c^{i+n}(X_{\mathbb{A}_n})$ is an extension of $\mathbb{Q}(i)$ by $\mathbb{Q}(i+1)$, hence has dimension 2.

Then the Künneth formula would imply that $H_c^{i+n+1}(Y_{\mathbb{A}_n})$ has dimension at least 2, which is absurd, as $H_c^{i+n+1}(Y_{\mathbb{A}_n})$ is $\mathbb{Q}(i)$ by Prop. 3.8. ∎

3.3 Smoothness

Let us prove that the varieties $X_n(\alpha)$ are smooth for generic α. Recall that α is assumed to be invertible.

Proposition 3.10 *For n even, $X_n(\alpha)$ is smooth.*

For n odd and $\alpha \neq (-1)^{(n+1)/2}$, $X_n(\alpha)$ is smooth.

For n odd and $\alpha = (-1)^{(n+1)/2}$, $X_n(\alpha)$ has a unique singular point: $x_i = x'_i = 0$ for odd i and $x_i = x'_i = -(-1)^{(n+i)/2}$ for even i.

Proof. The proof is by induction on n. The statement is clear if $n = 0, 1$.

Assume that there is a singular point on $X_{n+2}(\alpha)$.

The equations defining a singular point on $X_{n+2}(\alpha)$ are the n equations of $X_{n+2}(\alpha)$ together with the vanishing of all minors of rank n of the $2n \times n$ matrix $M_{n+2}(\alpha)$ of partial derivatives of these n equations with respect to variables $x_1, \dots, x_n, x'_1, \dots, x'_n$. This matrix $M_{n+2}(\alpha)$ looks as follows:

$$\begin{bmatrix} x'_1 & -\alpha & 0 & \dots & 0 & x_1 & 0 & & \dots & 0 \\ -x_3 & x'_2 & -x_1 & 0 & & 0 & x_2 & 0 & & \\ 0 & -x_4 & x'_3 & -x_2 & 0 & 0 & 0 & x_3 & 0 & \\ & \ddots & \ddots & \ddots & & & & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & -1 & x'_n & 0 & \dots & & 0 & x_n \end{bmatrix} \tag{19}$$

Let us distinguish 2 cases and some sub-cases.

First case : $x_1 = 0$.

Using the equations, this hypothesis implies that $x_2 = -1/\alpha$ and $x'_2 = -\alpha$.

Assume first that $x'_1 = 0$. Then the vanishing of all minors of the matrix $M_{n+2}(\alpha)$ reduces to the vanishing of all minors of the matrix $M_n(-1/\alpha)$ (with a shift of indices by 2). So the singular point gives, by restriction to coordinates $(x_i, x'_i)_{i \geq 3}$, a singular point on $X_n(-1/\alpha)$.

If $n + 2$ is even or $n + 2$ is odd and $\alpha \neq (-1)^{(n+2+1)/2}$, this is absurd by induction hypothesis.

If $n+2$ is odd and $\alpha = (-1)^{(n+2+1)/2}$, there is only one solution by induction hypothesis: $x_i = x'_i = 0$ for odd $i \geq 3$ and $x_i = x'_i = -(-1)^{(n+i)/2}$ for even $i \geq 2$. It is readily checked that the point $(x_i, x'_i)_{i\geq 1}$ is indeed a singular point on $X_{n+2}(\alpha)$.

Assume on the contrary that $x'_1 \neq 0$. Then the vanishing of all minors of the matrix $M_{n+2}(\alpha)$ reduces to the vanishing of all minors of an extended matrix which is made of $M_n(-1/\alpha)$ plus one more column on the left:

$$\begin{bmatrix} -x_4 & x'_3 & -x_2 & 0 & \dots & 0 & x_3 & 0 & \dots & 0 \\ 0 & -x_5 & x'_4 & -x_3 & 0 & & 0 & x_4 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & & & & \ddots & \ddots \\ 0 & 0 & \dots & 0 & -1 & x'_n & 0 & \dots & 0 & x_n \end{bmatrix} \tag{20}$$

In particular, by restriction to coordinates $(x_i, x'_i)_{i\geq 3}$, we obtain a singular point on $X_n(-1/\alpha)$.

If $n+2$ is even or $n+2$ is odd and $\alpha \neq (-1)^{(n+2+1)/2}$, this is absurd by induction hypothesis.

If $n+2$ is odd and $\alpha = (-1)^{(n+2+1)/2}$, there is only one solution by induction hypothesis: $x_i = x'_i = 0$ for odd $i \geq 3$ and $x_i = x'_i = -(-1)^{(n+i)/2}$ for even $i \geq 2$. One can then check that the extended matrix has a non-vanishing minor at this point, and therefore $(x_i, x'_i)_{i\geq 1}$ is not a singular point on $X_{n+2}(\alpha)$. This is absurd.

Second case : $x_1 \neq 0$. Then the vanishing of all minors of the matrix $M_{n+2}(\alpha)$ reduces to the vanishing of all minors of an an extended matrix which is made of $M_{n+1}(x_1)$ (with a shift of indices by 1) plus one more column on the left:

$$\begin{bmatrix} -x_3 & x'_2 & -x_1 & 0 & \dots & 0 & x_2 & 0 & \dots & 0 \\ 0 & -x_4 & x'_3 & -x_2 & 0 & & 0 & x_3 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & & & & \ddots & \ddots \\ 0 & 0 & \dots & 0 & -1 & x'_n & 0 & \dots & 0 & x_n \end{bmatrix} \tag{21}$$

In particular, by restriction to coordinates $(x_i, x'_i)_{i\geq 2}$, we obtain a singular point on $X_{n+1}(x_1)$.

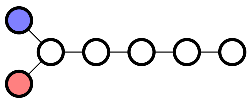
If $n+2$ is odd, this is absurd by induction hypothesis.

If $n+2$ is even, then there is only one possible solution by induction hypothesis: $x_i = x'_i = 0$ for even $i \geq 2$ and $x_i = x'_i = -(-1)^{(n+1+i)/2}$ for odd $i \geq 3$. One can then check that the extended matrix has a non-vanishing minor at this point, which is therefore not a singular point on $X_{n+2}(\alpha)$.

■

4 Type $\mathbb{D}$

We will now consider the Dynkin diagrams of type $\mathbb{D}$.



In type $\mathbb{D}_n$, the cluster algebra is generated by n cluster variables $x_1, x_2, x_3, \dots, x_n$ (which form a cluster), the n adjacent cluster variables $x'_1, x'_2, x'_3, \dots, x'_n$ and n coefficient variables $\alpha_1, \dots, \alpha_n$ with the following relations:

$$\begin{aligned}
x_1 x'_1 &= 1 + \alpha_1 x_3, \\
x_2 x'_2 &= 1 + \alpha_2 x_3, \\
x_3 x'_3 &= 1 + \alpha_3 x_1 x_2 x_4, \\
x_4 x'_4 &= 1 + \alpha_4 x_3 x_5, \\
&\dots \\
x_{n-1} x'_{n-1} &= 1 + \alpha_{n-1} x_{n-2} x_n, \\
x_n x'_n &= 1 + \alpha_n x_{n-1}.
\end{aligned}$$

Let us call $X_{\mathbb{D}_n}(\alpha_1, \dots, \alpha_n)$ this variety. As before, we assume that the α_i are invertible.

Proposition 4.1 *If n is even, then $X_{\mathbb{D}_n}(\alpha_1, \dots, \alpha_n) \simeq X_{\mathbb{D}_n}(\alpha, \beta, 1, \dots, 1)$, for some α, β depending on the α_i.*

If n is odd, then $X_{\mathbb{D}_n}(\alpha_1, \dots, \alpha_n) \simeq X_{\mathbb{D}_n}(\alpha, 1, \dots, 1)$, for some α depending on the α_i.

Proof. This is obtained by applying Proposition 2.4 to the partial domino tiling avoiding the first two vertices (even case) or to the partial domino tiling avoiding only the first vertex (odd case). ■

Let us introduce some notation for the number of points of these varieties over finite fields. If n is odd, we will denote $N_{\mathbb{D}_n}(\alpha)$ the number of points of $X_{\mathbb{D}_n}(\alpha, 1, \dots, 1)$. If n is even, we will denote $N_{\mathbb{D}_n}(\alpha, \beta)$ the number of points of $X_{\mathbb{D}_n}(\alpha, \beta, 1, \dots, 1)$.

Proposition 4.2 *If n is odd and $\alpha \neq 1$, then*

$$N_{\mathbb{D}_n}(\alpha) = q^n - 1. \tag{22}$$

If n is odd, then

$$N_{\mathbb{D}_n}(1) = q^n - 1 + q^2 \frac{q^{n-1} - 1}{q^2 - 1}. \tag{23}$$

If n is even, $\alpha \neq \beta$, $\alpha \neq (-1)^{n/2}$ and $\beta \neq (-1)^{n/2}$, then

$$N_{\mathbb{D}_n}(\alpha, \beta) = (q^{n/2} - 1)^2. \tag{24}$$

If n is even, and $\alpha = \beta$ differs from $(-1)^{n/2}$, then

$$N_{\mathbb{D}_n}(\alpha, \alpha) = (q^{n/2} - 1)^2 + q^2 \frac{(q^{(n-2)/2} - 1)(q^{n/2} - 1)}{q^2 - 1}. \tag{25}$$

If n is even, and $\alpha \neq \beta$ and $\alpha = (-1)^{n/2}$, then

$$N_{\mathbb{D}_n}((-1)^{n/2}, \beta) = (q^{n/2} - 1)^2 + (q - 1) q^{n/2}. \tag{26}$$

If n is even, and $\alpha = \beta = (-1)^{n/2}$, then $N_{\mathbb{D}_n}((-1)^{n/2}, (-1)^{n/2})$ equals

$$(q^{n/2} - 1)^2 + 2(q - 1) q^{n/2} + q^2 \frac{(q^{(n-2)/2} - 1)(q^{n/2} - 1)}{q^2 - 1} + q^{(n+2)/2}. \tag{27}$$

Proof. The proof uses leaf-removal (Proposition 2.6) and knowledge of type $\mathbb{A}$.

If $n = 3$, one has $X_{\mathbb{D}_3}(\boldsymbol{\alpha}) \simeq X_{\mathbb{A}_3}(\boldsymbol{\alpha})$ and the statement follows from type $\mathbb{A}$.

If $n \geq 5$ is odd, let us remove the leaf with value α. One gets, using Lemma 2.2 to compute the rightmost term,

$$N_{\mathbb{D}_n}(\alpha) = qN_{\mathbb{A}_1}(-1/\alpha)N_{\mathbb{A}_{n-3}} + (q-1)N_{\mathbb{A}_{n-1}}. \tag{28}$$

According to results in type $\mathbb{A}$, one therefore has to distinguish the case $\alpha = 1$. One then compute using Prop. 3.2.

If n is even, let us remove the leaf with value α. One gets

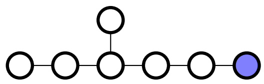
$$N_{\mathbb{D}_n}(\alpha, \beta) = qN_{\mathbb{A}_1}(-\beta/\alpha)N_{\mathbb{A}_{n-3}}(-1/\alpha) + (q-1)N_{\mathbb{A}_{n-1}}(\beta). \tag{29}$$

According to Prop. 3.2, one has to distinguish according to three alternatives: $\beta = \alpha$ or not, $\alpha = (-1)^{n/2}$ or not, and $\beta = (-1)^{n/2}$ or not. One can also use the symmetry exchanging α and β. In each case, one can compute the result using Prop. 3.2. ∎

Remark 4.3 *One may wonder, in type $\mathbb{D}_n$ with n odd, if the homotopy type of $X_{\mathbb{D}_n}(\alpha)$ (for $\alpha \neq 1$) is that of a sphere.*

5 Type $\mathbb{E}$

We will now consider the Dynkin diagrams of type $\mathbb{E}$.



Using the general definition given for trees, one can introduce varieties $X_{\mathbb{E}_6}(\boldsymbol{\alpha})$, $X_{\mathbb{E}_7}(\boldsymbol{\alpha})$ and $X_{\mathbb{E}_8}(\boldsymbol{\alpha})$ depending on invertible parameters $\boldsymbol{\alpha}$.

Proposition 5.1 *Every variety $X_{\mathbb{E}_6}(\boldsymbol{\alpha})$ is isomorphic to the variety $X_{\mathbb{E}_6}(1, \dots, 1)$. Every variety $X_{\mathbb{E}_7}(\boldsymbol{\alpha})$ is isomorphic to the variety $X_{\mathbb{E}_7}(1, \dots, 1, \alpha)$, where α is the value on the last vertex on the long branch of $\mathbb{E}_7$. Every variety $X_{\mathbb{E}_8}(\boldsymbol{\alpha})$ is isomorphic to the variety $X_{\mathbb{E}_8}(1, \dots, 1)$.*

Proof. This follows from Proposition 2.4, using appropriate domino tilings. ∎

Let us introduce some notation for the number of points of these varieties over finite fields. We will denote $N_{\mathbb{E}_6}$ the number of points of $X_{\mathbb{E}_6}(1, \dots, 1)$. and $N_{\mathbb{E}_8}$ the number of points of $X_{\mathbb{E}_8}(1, \dots, 1)$. We will denote $N_{\mathbb{E}_7}(\alpha)$ the number of points of $X_{\mathbb{E}_7}(1, \dots, 1, \alpha)$ where α is the value on the last vertex on the long branch of $\mathbb{E}_7$.

Proposition 5.2 *The number of points are as follows:*

$$\begin{array}{rcl} N_{\mathbb{E}_6} & = & q^6 + q^4 + q^3 + q^2 + 1, \\ N_{\mathbb{E}_7}(\alpha) & = & q^7 + q^5 - q^2 - 1 \quad \textit{if} \quad \alpha \neq -1, \\ N_{\mathbb{E}_7}(-1) & = & q^7 + 2q^5 + q^3 - q^2 - 1, \\ N_{\mathbb{E}_8} & = & q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1. \end{array} \tag{30}$$

Proof. The proof uses leaf-removal (Proposition 2.6) and the knowledge of the numbers in type $\mathbb{A}$.

In the case of $\mathbb{E}_6$, let us remove the leaf of the shortest branch. One gets, using Lemma 2.2 to compute the second term,

$$N_{\mathbb{E}_6} = qN_{\mathbb{A}_2}N_{\mathbb{A}_2} + \sum_{\beta\in\mathbb{F}_q^*} N_{\mathbb{A}_5}(\beta). \tag{31}$$

One can compute this using Prop. 3.2 and Lemma 3.3.

In the case of $\mathbb{E}_7$, let us remove the leaf of the shortest branch. One gets

$$N_{\mathbb{E}_7}(\alpha) = qN_{\mathbb{A}_2}N_{\mathbb{A}_3}(-1/\alpha) + (q-1)N_{\mathbb{A}_6}. \tag{32}$$

Therefore, by Prop. 3.2 applied to $\mathbb{A}_3$, one has to separate the case $\alpha = -1$. One can compute the different results using Prop. 3.2.

In the case of $\mathbb{E}_8$, let us remove the leaf of the shortest branch. One gets (using Lemma 2.2 to compute the second term)

$$N_{\mathbb{E}_8} = qN_{\mathbb{A}_2}N_{\mathbb{A}_4} + \sum_{\beta\in\mathbb{F}_q^*} N_{\mathbb{A}_7}(\beta). \tag{33}$$

One can compute this using Prop. 3.2 and Lemma 3.3.

■

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